Approximations for the Nonlinear Self-Channel Interference of Channels With Rectangular Spectra

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***Abstract*— The Gaussian noise (GN) model, in which the fiber nonlinearity is modeled as an additive GN process, has been recently shown in the literature to be accurate for uncompensated coherent systems. Nevertheless, it does not have an exact analytical solution requiring analytical approximations to be made. Herein, we propose a new means of approximating the nonlinear self-channel interference (SCI) in the GN model, for the case of ideal Nyquist WDM channels that have rectangular spectra, bandlimited to the Nyquist bandwidth. We begin by introducing the method to estimate the peak power spectral density of the nonlinear interference before applying it to calculating the total SCI noise of a channel. The analytical solution is compared with the previously reported approximation and the exact numerical solution, to quantify the approximation error. The proposed approximation is accurate to within 0.3 dB of the GN model as the symbol rate is varied from 10 to 100 GBd. Finally, we demonstrate that for a superchannel, the total nonlinear interference for the central channel can be approximated to within 0.3 dB for three or more channels.**

***Index Terms*— Approximation methods, GN model, optical fiber communication.**

I. Introduction

As optical transmission systems move toward employing digital coherent transceivers fundamental changes occur regarding the interaction of the nonlinear induced penalty in the signal. For these systems, since all of the chromatic dispersion compensation is performed in the digital domain, the signal amplitude rapidly becomes Gaussian in nature. This in turn allows previously considered analytical techniques such as the Gaussian noise model, proposed by Splett *et. al*. in 1993 [1], to be revisited. The GN model has recently shown to give excellent agreement with numerical solutions for uncompensated systems [2], [3]. Herein we focus on the results from Poggiolini *et. al*. and adopt their taxonomy in which the single channel nonlinearity due to the Kerr effect is referred to as self-channel interference (SCI) [4]. In this letter we present an equivalent area method of integration resulting in accurate approximations to the SCI. We also derive an approximation for the total nonlinear interference of the central channel of a superchannel allowing a lower bound on the WDM nonlinear performance to be obtained and contrasted with the upper bound obtained by considering the SCI of the individual channel alone.

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II. Nyquist Channels With Rectangular Spectra

Within this letter we will focus on ideal "Nyquist WDM" channels that satisfy the Nyquist criterion, having a rectangular spectra of width equal to the symbol rate [4]. As such we define the rectangular function rect (x) which we denote for compactness as $\Pi(x)$ given by

$$\mathrm{rect}(x) = \Pi(x) = \begin{cases} 1, & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

from which it follows using the Gaussian noise model [4] using frequencies normalized to the symbol rate, that the peak power spectral density $G_{\mathrm{SCI}}(0)$ and the self-channel interference noise σ^2_{SCI} over a single span of length L_s are given by

$$G_{\mathrm{SCI}}(0) = \frac{16P^3\gamma^2}{27B} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\Pi(x)\Pi(y)\Pi(x+y)\mathrm{d}x\mathrm{d}y \tag{2}$$

and

$$\sigma^2_{\mathrm{SCI}} = \frac{16P^3\gamma^2}{27} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\Pi(x+y+z)\Pi(z) \times\Pi(x+z)\Pi(y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{3}$$

where P is the channel power, B is the bandwidth of the signal, corresponding to the symbol rate for a Nyquist channel and $K(x, y)$ is a normalized kernel function related to the four wave mixing efficiency given by

$$K(x,y) = \frac{L^2_{\mathrm{eff}}\alpha^2}{\alpha^2 + 16\pi^4\beta_2^2B^4x^2y^2} \tag{4}$$

being valid for a span loss of 7 dB or greater [4], where α and β_2 are the attenuation and dispersion coefficients of the fiber and $L_{\mathrm{eff}} = (1-\exp(-\alpha L_s))/\alpha$ is the effective length of the fiber. The aim of this letter is to approximate these integrals to provide an analytical expressions for the $G_{\mathrm{SCI}}(0)$ and σ^2_{SCI}.

III. Equivalent Area Approximation of $G_{\mathrm{SCI}}(0)$

We begin by turning our attention to evaluating $G_{\mathrm{SCI}}(0)$ in order to introduce the equivalent area approximation. We begin by noting that for low symbol rates $K(x, y) \approx L^2_{\mathrm{eff}}$

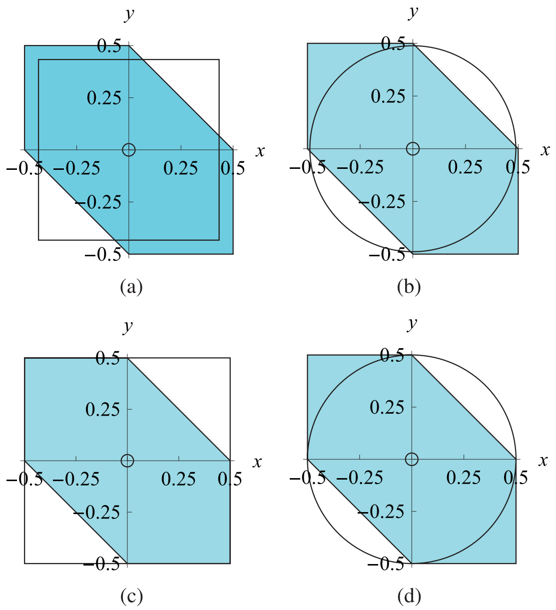

Fig. 1. Shaded integration region: polygon defined by $\Pi(x)\Pi(y)\Pi(x+y)$. In addition, square and circular integration regions are shown having an equivalent area and the same maximal dimensions as the polygon. (a) Square equivalent area. (b) Circular equivalent area. (c) Square maximal domain. (d) Circular maximal domain.

and as such the integral becomes trivial, given that region of integration, defined by $\Pi(x)\Pi(y)\Pi(x+y)$ has an area of 0.75 squared units. While the region of integration defined by $\Pi(x)\Pi(y)\Pi(x+y)$ is straightforward when $K(x,y)$ is constant, for the general case where $K(x,y)$ is not constant it is more challenging. As such we propose to replace exact region of integration by a simpler region such as a circular or square one like those illustrated in Fig. 1, the dimensions of which are chosen to also give an area of 0.75 squared units. By making the region of integration the same area for both cases ensures that the approximate solution converges to the exact solution for low symbol rates.[1]

Hence defining the region R that encloses the equivalent area $A_{\text{eq}} = 0.75$, we have our equivalent area approximation for $G_{\text{SCI}}(0)$ as

$$G_{\text{SCI}}(0) = \frac{16P^3\gamma^2}{27B}\iint_R K(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{5}$$

We propose two alternative regions R to bound the area A_{eq} in addition to the two maximal regions[2] also proposed in [4].

1) A square equivalent area region having vertices at $(x,y) = (\pm\sqrt{3}/4, \pm\sqrt{3}/4)$, illustrated in Fig. 1(a).
2) A circular equivalent area region centered at the origin of radius $\frac{1}{2}\sqrt{3/\pi}$, illustrated in Fig. 1(b).
3) A square maximal domain region having vertices at $(x,y) = (\pm 1/2, \pm 1/2)$, illustrated in Fig. 1(c).
4) A circular maximal domain region centered at the origin of radius $1/2$, illustrated in Fig. 1(d).

[1]This condition more being for mathematical convenience rather than a desire to have an exact solution for very low symbol rates where the amplitude may not be Gaussian in contravention of the key assumption of the GN model.

[2]The circular and square approximations were independently investigated prior to the publication of [4] which appeared while finalizing this manuscript. The regions proposed in Fig. 24 of [4] have a maximal domain with respect to the integrand $K(x,y)$ since $K(x,y)$ is concentrated on the x and y axes.

A. Square Equivalent Area Approximation for $G_{\text{SCI}}(0)$

If we define the inverse tangent integral $\mathrm{Ti}_2(x)$ as

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\arctan(u)}{u}\mathrm{d}u \tag{6}$$

then the square region may be integrated exactly as

$$\begin{aligned} G_{\text{SCI}}(0) &= \frac{16P^3\gamma^2 L_{\text{eff}}^2}{27B}\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}}\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{\mathrm{d}x\,\mathrm{d}y}{1+\left(\frac{4\pi^2|\beta_2|B^2xy}{\alpha}\right)^2} \\ &= \frac{16P^3\gamma^2 L_{\text{eff}}^2\alpha}{27B^3\pi^2|\beta_2|}\mathrm{Ti}_2\left(\frac{3\pi^2|\beta_2|B^2}{4\alpha}\right) \end{aligned} \tag{7}$$

where $\mathrm{Ti}_2(x)$ may be evaluated to an accuracy of less than 0.34% using the approximation detailed in the appendix.

B. Circular Equivalent Area Approximation for $G_{\text{SCI}}(0)$

For the circular approximation, we first convert to polar co-ordinates with $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and then exploit the four fold symmetry with regards to θ such that

$$\begin{aligned} G_{\text{SCI}}(0) &= \frac{64P^3\gamma^2 L_{\text{eff}}^2}{27B}\int_0^{\frac{1}{2}\sqrt{\frac{3}{\pi}}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r\,\mathrm{d}\theta\,\mathrm{d}r}{1+\left(\frac{2\pi^2|\beta_2|B^2r^2}{\alpha}\sin(2\theta)\right)^2} \\ &= \frac{32\pi P^3\gamma^2 L_{\text{eff}}^2}{27B}\int_0^{\frac{1}{2}\sqrt{\frac{3}{\pi}}} \frac{r\,\mathrm{d}r}{\sqrt{1+\left(\frac{2\pi^2|\beta_2|B^2r^2}{\alpha}\right)^2}} \\ &= \frac{8P^3\gamma^2 L_{\text{eff}}^2\alpha}{27B^3\pi|\beta_2|}\operatorname{arsinh}\left(\frac{3\pi|\beta_2|B^2}{2\alpha}\right) \end{aligned} \tag{8}$$

where the integrals are readily integrated via standard substitutions.

C. Assessing the Accuracy of the Approximations for $G_{\text{SCI}}(0)$

To assess the accuracy of the approximations, we calculate the peak noise power spectral density for the exact expression and the approximation for a single channel, centered at 1550 nm propagating over single mode fiber (SMF), with chromatic dispersion of $\mathrm{D} = 16.7\ \mathrm{ps\cdot nm^{-1}\cdot km^{-1}}$, attenuation of $0.22\ \mathrm{dB\cdot km^{-1}}$, giving $\alpha = 0.0507\ \mathrm{km^{-1}}$ and nonlinear coefficient $\gamma = 1.3\ \mathrm{km^{-1}\cdot W^{-1}}$. We then define the approximation error in decibels, which not only has the benefit of being independent of the effective length, launch power and the number of spans,[3] but is more closely aligned with system performance.

As expected, for low symbol symbol rates in Fig. 2, the error for the equivalent area approximations are close to zero.

[3]From [4], the noise accumulation scales as $N_s^{1+\epsilon}$ where N_s is the number of spans and $\epsilon \in [0,1)$. For a fixed value of ϵ, such as $\epsilon = 0$ corresponding to incoherent addition, the scale factor $N_s^{1+\epsilon}$ does not affect the relative error and hence the approximation error when using a decibel scale.

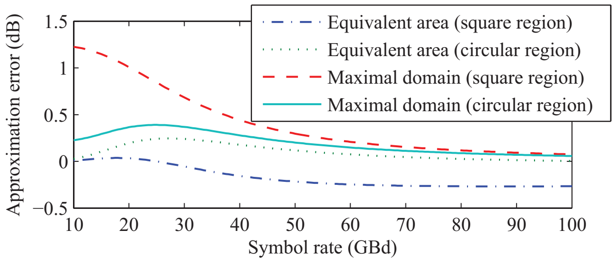

Fig. 2. Approximation error in $G_{SCI}(0)$ for various regions of integration.

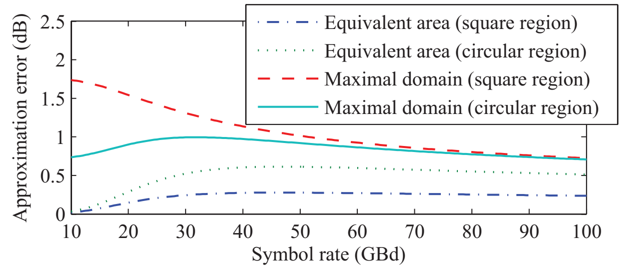

Fig. 3. Approximation error in σ^2_{SCI} for various regions of integration.

In contrast for the maximal domains, by expanding the approximations for small B, the expected error is $10\log_{10}(\pi/3) = 0.2$ dB and $10\log_{10}(4/3) = 1.2$ dB for the circular and square approximation regions respectively, in agreement with that observed. Of particular interest is the worst case approximation error, which for the approximation proposed herein is 0.2 dB using the circular equivalent area, in contrast to the square equivalent area approximation which gives an extremal error of −0.3 dB.

IV. Equivalent Area Approximation of σ^2_{SCI}

The total noise variance σ^2_{SCI} for Nyquist channels with rectangular spectra is given by (3) which again is problematic to integrate. However as noted in [3] one means of approximating σ^2_{SCI} is the locally white approximation such that

$$\sigma^2_{SCI} = \int_{-B/2}^{B/2} G_{SCI}(f)\mathrm{d}f \approx B \times G_{SCI}(0) \tag{9}$$

where $G_{SCI}(f)$ is the true spectral variation of the power spectral density. Inspired by this and our equivalent area approximation for $G_{SCI}(0)$ we consider an equivalent area approximation for σ^2_{SCI}, such that

$$\begin{aligned}\sigma^2_{SCI} &= B \times \frac{16P^3\gamma^2}{27B}\iint_R K(x,y)\mathrm{d}x\mathrm{d}y\\ &= \frac{16P^3\gamma^2}{27}\iint_R K(x,y)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{10}$$

where R is a region of equivalent area A_{eq}. Hence as before we begin by noting that for low symbol rates $K(x,y) \approx L^2_{eff}$ such the integral $\iint_R K(x,y)\mathrm{d}x\mathrm{d}y = L^2_{eff}A_{eq}$, where A_{eq} is the equivalent area which is given by

$$\begin{aligned}A_{eq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Pi(x+y+z)\Pi(z)\\ &\qquad\times\Pi(x+z)\Pi(y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \frac{2}{3}\end{aligned} \tag{11}$$

We propose two alternative regions R to bound the area A_{eq} being:

1) a square region having vertices at $(x,y) = (\pm 1/\sqrt{6}, \pm 1/\sqrt{6})$;
2) a circular region centered at the origin of radius $\sqrt{2/3}/\sqrt{\pi}$.

A. Square Equivalent Area Approximation for σ^2_{SCI}

Similar to the case for $G_{SCI}(0)$ the square region may be integrated exactly as

$$\sigma^2_{SCI} = \frac{16P^3\gamma^2L^2_{eff}\alpha}{27B^2\pi^2|\beta_2|}\mathrm{Ti}_2\left(\frac{2\pi^2|\beta_2|B^2}{3\alpha}\right) \tag{12}$$

where $\mathrm{Ti}_2(x)$ is given by (6) which may be evaluated using the approximation detailed in the appendix.

B. Circular Equivalent Area Approximation for σ^2_{SCI}

We again convert to polar co-ordinates with $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and then exploit the four fold symmetry of θ such that

$$\sigma^2_{SCI} = \frac{8P^3\gamma^2L^2_{eff}\alpha}{27B^2\pi|\beta_2|}\mathrm{arsinh}\left(\frac{4\pi|\beta_2|B^2}{3\alpha}\right) \tag{13}$$

As can be seen from Fig. 3, the maximum error for the equivalent area method is 0.3 and 0.6 dB for the square and circular region respectively.[4] For the maximal domains proposed by Poggiolini the error for small B can be obtained by expanding the approximations giving $10\log_{10}(3\pi/8) = 0.7$ dB and $10\log_{10}(3/2) = 1.8$ dB for the circular and square approximation regions respectively, in agreement with that observed.

V. Extension to Superchannels

One of the key benefits of using Nyquist channels with rectangular spectra is that a superchannel may be formed from several contiguous lower bandwidth channels with the resulting superchannel also having a rectangular spectra. Naturally if the whole superchannel is considered as one channel the results reported are equally applicable to the overall superchannel, albeit P becomes the total power of the superchannel and B the total bandwidth of the superchannel. At present however the channels are considered independently with the performance of the central channel for which the nonlinear interference is maximum being a key measure.[5] To calculate

[4]For different fiber types such as pure silica core fiber (PSCF) and non-zero dispersion shifted fiber (NZDSF), using the parameters from [3] the maximum error is unchanged, albeit the location of the maximum error is shifted.

[5]The nonlinear interference includes both self-channel interference of the central channel and the nonlinear cross channel interference from other channels within the superchannel. We note that a superchannel comprising of N channels with B corresponding the the EDFA window, giving $B \approx 5$ THz and $P = NP_{max}$ provides a lower bound on the WDM performance, where P_{max} is the maximum channel power.

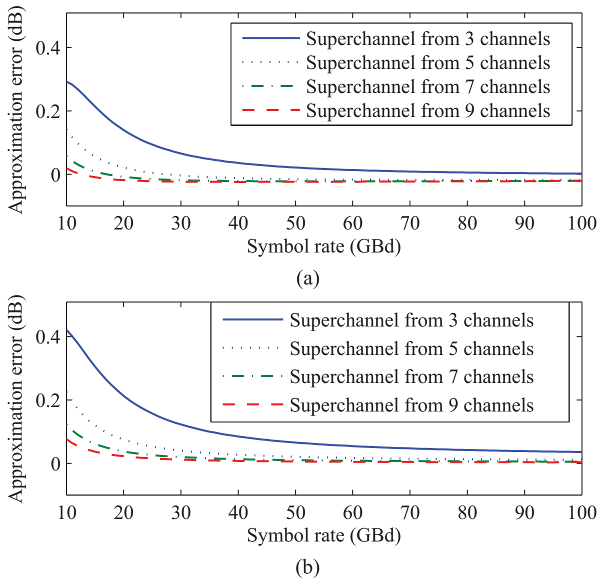

Fig. 4. Difference between the approximate and exact values of the nonlinear interference (including both single channel and WDM nonlinearities) of the central channel of an N channel superchannel. (a) White noise approximation with equivalent area (circular region). (b) White noise approximation with maximal domain (circular region).

the nonlinear interference for the central channel, one commonly used approximation is the locally white approximation, such that $\sigma^2_{NLI,c} \approx G_{\mathrm{SCI}}(0)B/N$, where N is the number of channels forming the superchannel occupying total bandwidth B. Using the Gaussian noise model the exact value for this is given by

$$\sigma^2_{NLI,c} = \frac{16P^3\gamma^2}{27N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\Pi(x+y+z)\Pi(zN) \times \Pi(x+z)\Pi(y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \quad (14)$$

being similar to the (3) with $\Pi(z) \to \Pi(zN)/N$ to extract the total noise for the central channel.

From Fig. 4(a) we note that for three or more channels, the white noise approximation, combined with the circular equivalent area approximation is accurate to within 0.3 dB, providing an accurate approximation for the performance of worst channel within a superchannel. We note however when the width of the superchannel is > 250 GHz there is a very small offset of -0.02 dB, which we note is the expected difference expected between the two approximations for superchannels of width approaching 1 THz.

To investigate this further we consider the maximal domain circular region, and from Fig. 4(b) we note that this small error of -0.02 dB is not present when using this approximation, albeit that the maximum error is higher for the cases considered. Nevertheless this validates the approach taken by Poggiolini, since it provides an excellent approximation when the superchannel bandwidth exceeds 250 GHz.

VI. Conclusion

The equivalent area method outlined in this letter enables accurate analytical approximations to the solution of the Gaussian noise model to be obtained. For the peak noise density the circular approximation proves to be the more accurate approximation being accurate to within 0.2 dB of the exact result, compared to 0.3 dB for the square approximation region. In contrast for the case of the nonlinear self-channel interference the square region is accurate to 0.3 dB, in contrast to 0.6 dB obtained using the circular approximation region. Finally the applicability of the method to superchannels highlighted that the locally white assumption, combined with the circular equivalent area approximation allows the nonlinear interference of the central channel to be estimated to an accuracy of 0.3 dB or better for three or more channels.

Appendix: Approximation for $\mathrm{Ti}_2(x)$

The inverse tangent integral defined by (6) is not widely tabulated and as such we seek a simpler approximation to this function. We begin by noting that the asymptotic expansion for $\mathrm{Ti}_2(x)$ is given by

$$\mathrm{Ti}_2(x) \approx \frac{\pi}{2}\log(x) + \frac{1}{x} - \frac{1}{9x^3} \quad (15)$$

and based on the Maclaurin series for $\mathrm{Ti}_2(x)$ we propose the following approximation for $x \in [0, a)$ as

$$\mathrm{Ti}_2(x) \approx x - \left(\frac{x}{b}\right)^3 + \left(\frac{x}{c}\right)^5 \quad (16)$$

where b and c are obtained by matching both the value and derivative of the two approximations at $x = a$ with a chosen to minimize the maximum absolute approximation error. The resulting minmax approximation with $a = 1.5330$, $b = 2.1962$, $c = 2.4313$, has two turning points with the maximum error of $+0.34\%$ at $x = 0.6964$ and the minimum error of -0.34% at $x = 1.4495$ giving our approximation as

$$\mathrm{Ti}_2(x) = \begin{cases} x - \left(\frac{x}{2.1962}\right)^3 + \left(\frac{x}{2.4313}\right)^5 & x \leq 1.533 \\ \frac{\pi}{2}\log(x) + \frac{1}{x} - \frac{1}{9x^3} & x > 1.533 \end{cases} \quad (17)$$

The piecewise minmax approximation to $\mathrm{Ti}_2(x)$ given by (17) is not only accurate to within 0.34%, for all x but is also straightforward to implement computationally.

Acknowledgment

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