Design of Nonlinear Regenerative Transmission Systems with High Capacity

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ABSTRACT
We present the design of nonlinear regenerative communication channels that have capacity above the classical Shannon capacity of the linear additive white Gaussian noise channel. The upper bound for regeneration efficiency is found and the asymptotic behavior of the capacity in the saturation regime is derived.

Keywords: Shannon nonlinear channel capacity, signal regeneration, signal filtering, system design

1. INTRODUCTION

The seminal Shannon's result [1] for capacity of the linear additive white Gaussian noise (AWGN) channel: $C = B\log_2(1+S/N)$ is one of the most celebrated equations in the history of science and engineering. In practical communication channels that are well approximated by the linear AWGN channel the Shannon capacity can be approached very closely (by a fraction of dB) by using low density parity check codes and turbo codes. The linear AGWN channel is nowadays a textbook material and is virtually wholly understood. There is, however, a growing interest in studies of a capacity of more complex communication channels including nonlinear channels for which the limits have yet to be defined. The well-known and extremely important for practical applications example is the optical fibre channel that is inherently nonlinear due to intensity-dependent refractive index in silica at high enough signal powers and very small fibre core where signal is guided over long distances [2-11].

The definition of the Shannon capacity per unit bandwidth for arbitrary channel [1] involves maximizing the mutual information (MI) functional:

$$C = \max_{P_x} \int DxDyP_x P_{y|x} \log_2 \frac{P_{y|x}}{\int DxP_x P_{y|x}} \tag{1}$$

over all valid input probability distributions subject to power constraint $\int dxP_x(x)|x|^2 \le S$. Here statistical properties of the channel are given by the conditional input-output probability density function (PDF) $P(y|x)$. Thus, from the view point of the information theory there is no much difference between linear or nonlinear channels as long as PDF $P(y|x)$ is known. Yet, the modern information theory is mostly developed for linear channels, simply because it is often technically challenging to derive $P(y|x)$ for practical nonlinear channels. This reflects both difficulty of analysis of nonlinear systems with noise and the fact that there are varieties of nonlinear communication channels that hardly can be described by any single generic theory. An important new feature introduced by nonlinearity is a possibility of nonlinear filtering or signal regeneration. Whenever the nonlinear filter transformation has multiple fixed points, the consequent interleaving of the accumulating noise with nonlinear filter will produce effective suppression of the noise. In other words the nonlinear filter will attract a signal to the closest fixed points and suppress the effective signal diffusion caused by the noise.

2. MODEL

Here we introduce a new method – regenerative mapping for designing nonlinear information channels with capacity exceeding Shannon capacity for linear AWGN channel. We start from substantial expansion of analysis presented in [12] and comprehensively quantify improvement in the Shannon capacity that various nonlinear channels with regenerators can provide over the linear AWGN channel. Next we introduce a new type of channels with smooth nonlinear transfer functions that are principally distinct from a ”hard decision” regeneration considered in [12]. We stress that the proposed channel model is fundamentally different from Decode-and-Forward channel model and does not assume any decoder/encoder pair to be used with in-line elements. The proposed regenerative element acts like nonlinear filter on the stochastic signal distortions through an effective periodic potential creating attraction regions in the signal mapping. Consider the nonlinear regenerative channel model with R identical nonlinear filters placed in the transmission line. The signal transmission is distorted by stochastic process which is modeled as AWGN uniformly distributed along the line. The noise term incorporates the stochastic effects from different sources. Depending on the model application,

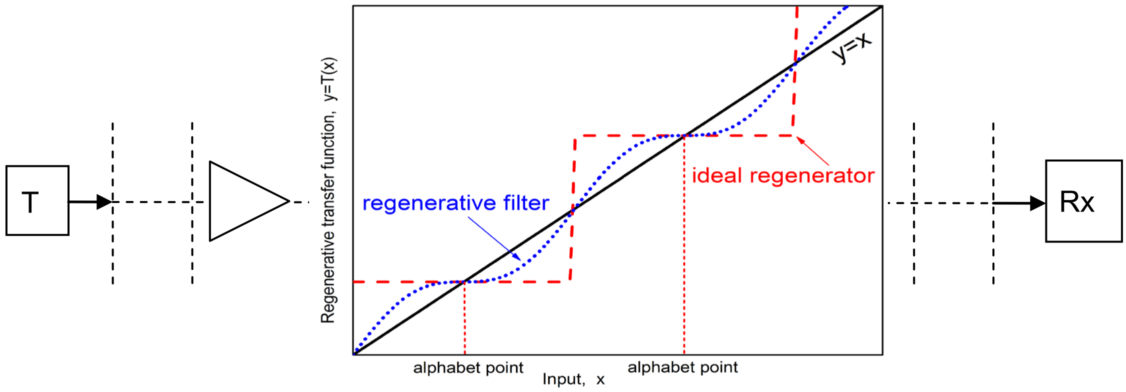

Figure 1. The regenerative channel scheme: nonlinear filters are placed equidistantly along the transmission line. The nonlinear filtering transfer function (TF) is plotted for the ideal regenerator (dashed line) and for the continuous mapping $y = x + \alpha \sin(\beta x)$ (dotted line).

the stochastic process can be considered as noise that adds up to signal during transmission through the media. This term also includes an additive noise that is originated from the nonlinear device itself. Here we develop the general mathematical method for constructing and optimizing the set of transfer functions $y = T(x)$ (see *e.g.* Fig. 1) that can increase nonlinear transmission system capacity beyond the capacity of the linear AWGN channel. The Shannon capacity (see eq. (1)) of the considered systems is a function of signal to noise ratio (SNR), the number of nonlinear filters R and the parameters of nonlinear mapping. SNR is defined here as the ratio of the input signal power S to the noise added linearly to the signal during transmission at each node $k = 1, \ldots, R$:

$$SNR = \frac{S}{N}, \qquad N = \sum_{k=1}^{R} N_k$$

Note that in the nonlinear communication system due to the mixing of signal with noise during propagation the definition of in-line SNR is a nontrivial issue. The introduced SNR has the meaning of the signal to noise ratio in the respective linear system in the absence of nonlinear in-line elements. This allows us to compare performance of the considered system with the corresponding linear AWGN channel having the same noise level. Evidently, the effect of noise squeezing is enhanced with the number of regenerators/nonlinear filters. Therefore, to evaluate the system performance improvement one needs to take into account the accumulative effect of nonlinear signal and noise transformation along the line. To quantify the overall effect, we study capacity of the source-destination transmission as a function of signal power ratio to the sum of powers of all added noise at the source-destination link.

We start the capacity analysis with the system of the ideal regenerators that is used further as a reference. In this case, we can derive analytical expressions for the maximum achievable regenerative capacity gain. Next, we introduce the general method of designing and optimization for the class of regenerative channels that guarantee capacity improvement without requirement of decoder/encoder pair. The considered model can be applied to signal propagation and regeneration in fiber-optic communication (see *e.g.* [3]-[9]).

2.1 Ideal regenerator

We start the analysis with the ideal regenerators that assign each transmitted symbol to the closest element of the given alphabet (see scheme in Fig. 1), the corresponding stepwise transfer function is plotted by the dashed red line). The step function defines a maximum regeneration capability and, consequently, determines the maximum achievable capacity gain due to nonlinear filtering. The conditional pdf of such a system is defined through matrix elements (see [2]):

$$P(y = x_k \mid x_l) = \int_{S_k} dx' P_C(x' \mid x_l) = W_{kl}, \tag{2}$$

where transition matrix W is defined as follows

$$W_{kl} = \frac{1}{2}(erf[\Delta_{kl}^{+}] - erf[\Delta_{kl}^{-}]), \quad \Delta_{kl}^{\pm} = (x_k + x_{k\pm1} - 2x_l)\sqrt{\frac{R}{8N}}$$

Due to Markovian nature of the stochastic system, the overall transition matrix after R regenerative segments reads as W^R. At low SNR range the channel is binary in each of n dimensions. Therefore, capacity is well approximated by the following expression:

$$\underline{C}_R = n[1 + m_+ \log_2(m_+) + m_\pm \log_2(m_\pm)]\,, \quad m_+ = \frac{1}{2^R}(1 + erf[\sqrt{R\rho/2}])^R\,, \quad m_\pm = 1 - m_+ \tag{3}$$

here denote SNR as ρ. As SNR increases, the closest neighbors distance reaches the optimal cell size, which is defined by the noise variance and the number of in-line regenerators, $d_{opt} = \sqrt{16\mathrm{N}\,\Omega[\mathrm{e}^2\mathrm{R}^2/(16\pi)]/\mathrm{R}}$, here Ω is the so-called Lambert function. Thus, at high SNR values the system is characterized by the optimal decision boundaries that are sufficiently large compared with the noise variance to suppress noise effectively. Therefore, with the growing signal power the amplitude distribution remains constant, whereas the Gaussian distribution is the optimal pdf for a fixed average energy constraint.

In the limit of high SNR and/or large number of regenerators, so that $\Delta = \sqrt{2\Omega[\mathrm{e}^2\mathrm{R}^2/(16\pi)]}$ >>1, the noise is sufficiently squeezed and the faulty decision occurs only between the nearest neighbors. With growing SNR one can observe a constant gap between the regenerative channel and linear AWGN channel capacities defined by the noise variance and the number of regenerators:

$$\Delta C_R = \log_2[\pi e R/(4\Delta)] + \frac{\mathrm{Re}^{-\Delta^2}}{\Delta\sqrt{\pi}}\log_2[\mathrm{Re}^{-\Delta^2}/(\Delta\sqrt{\pi})]\,, \quad \overline{C_R} = \log_2[1 + S/N)] + \Delta C_R \tag{4}$$

Thus, the maximum capacity gain due to regeneration (*i.e.* the maximum regeneration efficiency) is observed for the binary channel. This result emphasizes efficiency of a simple binary channel that might be important for practical design consideration. The minimum SNR value, when d_{opt} is achieved, defines the maximum capacity ratio to its linear analogue, *i.e.* $SNR_{opt} = d_{opt}^2/(4N)$. Also, at this SNR value two analytic formulas (eq. (3), (4)) can be interpolated to describe capacity at the full range of SNR values. The analytical approximations shown in the left panel of Fig. 2 by black lines demonstrate an excellent agreement with the result of numerical computations of the capacity gain for different numbers of regenerators. The figure demonstrates that regeneration allows substantially reduce noise impairments and, consequently, achieve capacity above the Shannon linear capacity. The maximum system improvement is observed at low SNR values – SNR_{opt}. The right panel of Fig. 2 shows capacity for quadrature amplitude modulation, QAM, and phase-shift keying, PSK, formats. Here the role of optimization is demonstrated: the optimal constellation choice depends on the SNR value and the number of the in-line regenerators. It is important to note, that with the non-optimal format one can get the resulting nonlinear capacity lower than the linear AWGN channel Shannon capacity (due to non-optimal value of the cell size), whereas the correct format choice can give dramatic gain in capacity.

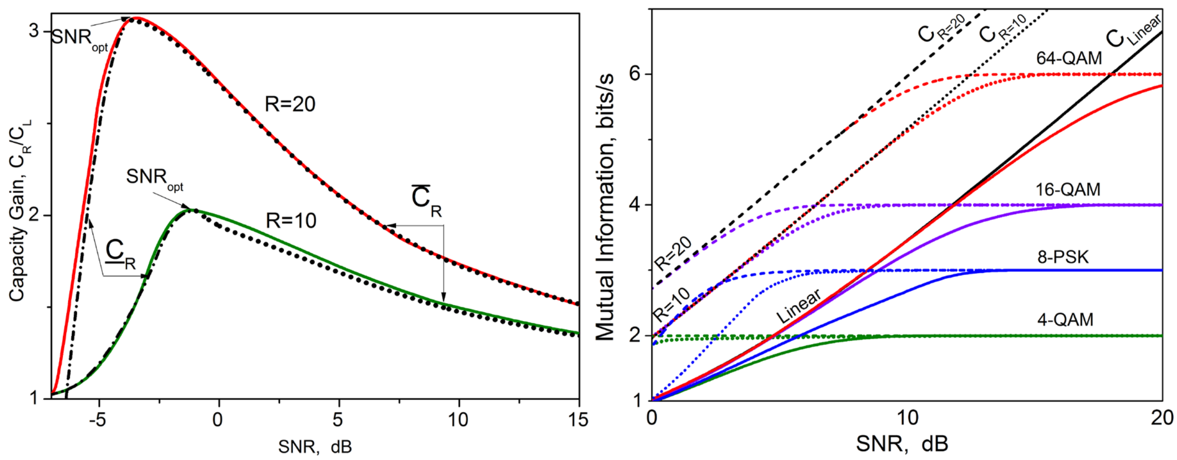

Figure 2. Left panel shows the capacity gain for the different number of regenerators. The analytical results shown by dash-dotted (equation (3)) and dotted (equation (4)) lines demonstrate an excellent agreement with numeric simulations (solid lines). Capacity compared to MI for discrete QAM and PSK formats (right panel).

2.2 General model for regenerative filters

The signal evolution in the nonlinear system with regenerative filters can be presented by the stochastic map:

$$y_k = T(y_{k-1}) + \eta_k\,, \qquad \text{with} \quad k = 1,\dots R \qquad \text{and} \qquad y_0 = x + \eta_0\,, \qquad y = y_R \tag{5}$$

Here k is the discrete spatial/temporal index and T is the transfer function of the regenerative filter (see the channel scheme in Fig. 1). The term η_k models AWGN (scheme is shown in Fig. 1). The nonlinear map has a set of special points that are optimal for nonlinear filtering. The sine-defined transformations result in the effective periodic potential, which creates attraction regions in the signal mapping without making the hard decision. Whereas the points are "attracted" to the alphabet, the alphabet itself should remain stable. These results in the following set of conditions imposed on the transfer function:

$$T(x^*) = x^*, \qquad T''(x^*) = 0, \qquad |T'(x^*)| \leq 1 \tag{6}$$

The first expression implies that the alphabet is defined by the stationary points x^* of the mapping. Next, the transfer function should change curvature at the alphabet, *i.e.* the alphabet point is the inflection point of the transfer function. The third expression reflects stability, so that the signal points distortion is effectively suppressed. When the first derivative is equal to zero, the alphabet is superstable.

In the limit of large SNR and/or large number of nonlinear filters all regenerative schemes tend to the asymptotic behaviour, when the gain gap between regenerative and linear AWGN channel capacity is constant. The saturation effect occurs when noise is squeezed to such level that the stochastic distortion is small and the shift takes place within the plateau area. Thus, the capacity gain of any optimized regenerative system tends to the asymptotic value defined above as $\overline{C_R}$.

3. CONCLUSIONS

We have developed the analytical model that proves the information capacity increase in stochastic systems with regenerative mapping. The gain is achieved by the noise squeezing due to introduced nonlinear filter design that creates attraction regions around the stable alphabet. We presented the design rules for implementation of nonlinear regenerative systems with increased channel capacity. The introduced classes of nonlinear devices can be used for construction of nonlinear communication channels with capacity exceeding the Shannon capacity of the linear AWGN channel. We anticipate that our results will lead to new insights into the Shannon capacity of nonlinear communications channels.

ACKNOWLEDGEMENTS

The support under the UK EPSRC Programme Grant UNLOC EP/J017582/1 is gratefully acknowledged

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